Convection in a porous medium with horizontal and vertical temperature gradients.

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Abstract.

The stability of convection in a horizontal porous layer subjected to horizontal as well as vertical temperature gradients is investigated. The boundaries are taken to be perfectly conducting and the horizontal temperature gradient is assumed to be small. The analysis shows that the critical Rayleigh number is always larger than for the ordinary Bénard problem in a porous medium. The preferred mode of disturbance is stationary, being longitudinal rolls, i.e. rolls having axes aligned in the direction of the basic flow. This particular mode minimizes the potential energy. Assuming that the initially preferred mode also dominates at supercritical Rayleigh numbers, a finite amplitude solution is obtained. The vertical heat flux is computed to second order. Compared with Bénard convection in a porous medium, the perturbation heat flux is diminished. The flux conducted through the boundaries is increased, however, so the total vertical heat flux is increased.

Nomenclature.

h	depth of porous medium;
d	characteristic grain diameter;
K	permeability of porous medium;
g	acceleration of gravity;
c_p	specific heat at constant temperature;
t	time;
$\vec{i},\vec{j},\vec{k}$	unit vectors;
$\vec{v}(=u,v,w)$	velocity vector;
$U(y)$	basic flow velocity;
T	temperature;
T_o	standard temperature;
ΔT	temperature difference between lower and upper plane;
p	pressure;
$P(x,y)$	pressure in the basic flow;
$\mathscr{H},H$	heat fluxes;
k,m	wave numbers in the x and z direction;
l_1,l_2	dimensionless lengths;
$\nabla^2\,(=\frac{\partial^2}{\partial y^2}+\nabla_1{}^2)$	Laplacian operator;
Δ	operator defined by (3.3);
L	operator defined by (3.14);
Re	Reynolds number;
Pr	Prandtl number ν/κ_m ;
Ra	Rayleigh number $Kg\gamma\Delta Th/\kappa_m\nu$;
A	amplitude of disturbance;
KE,F	defined by (4.2) and (4.4), respectively.

Greek letters

α	overall wave number;
$\alpha_2{}^2, \alpha_4{}^2$	defined by (3.22) and (3.26), respectively;
β	horizontal temperature gradient;
γ	coefficient of volume expansion;
δ	fraction of heat;
θ, Θ	temperatures;
$\kappa_m (= \lambda_m / (c_p \rho)_f)$	thermal diffusivity;
λ_m	thermal conductivity;
ν	kinematic viscosity;
ρ	density;
ρ_o	standard density;
σ	amplification factor of disturbance;
ψ	potential defined by (3.3).
ε	parameter defined by (5.3)

Subscripts

f	fluid;
m	solid-fluid mixture;
h	horizontal;
v	vertical.

Superscripts

$\hat{}$	perturbation quantities;
$\tilde{}$	y-dependent part of linear perturbations;
r	real part;
i	imaginary part;
c	critical.

1. Introduction.

Buoyancy driven convection in a porous medium has several important geophysical and technical applications. Thus, geothermal activities in certain areas of the world may be attributed to this phenomenon [1]. It also may be present in natural gas reservoirs [2]. Technically this phenomenon is important as it may occur in porous insulation of buildings, thereby increasing the loss of heat.

The present paper is concerned with free convection in a horizontal porous layer, where the ratio of height to length is small. When uniformly heated from below, this model has been investigated by several authors during the past thirty years or so. Especially in the last few years considerable efforts have been made in understanding this subject. Among the recent papers we mention [2-8], where also references to earlier works can be found.

In a physical problem, however, strictly uniform heating generally does not occur. Thus, horizontal as well as vertical temperature gradients will be present. For thin viscous layers this problem has motivated some previous investigations, [9-12] where various lateral heating conditions have been used. Most recently Weber [13] has made an analysis of this problem, assuming that the temperature varies linearly along the boundaries, while the vertical temperature difference is kept constant. In the present paper this model is applied to convection in a porous medium.

Owing to the similarity between convection in a fluid with infinite Prandtl number and porous convection, several conclusions can be obtained from [13]. As in this paper the lateral temperature variation produces a horizontal shear flow. This becomes unstable to infinitesimal perturbations when the vertical temperature difference is sufficiently increased. Owing to the existence of the basic flow, a unique perturbation pattern is predicted from linear theory. This is longitudinal rolls, or rolls with axes aligned in the direction of the basic flow.

The equation for momentum transfer in a porous medium does not contain convective terms. Hence shear instability can not occur. This conforms to the ideas in [14], where it is proved that convection in a porous vertical slab is stable. Then the mechanism of instability in the present problem must be of thermal origin.

In the last part of the paper we extend the analysis to the nonlinear regime. Considering the initially preferred mode, we obtain a finite amplitude solution. The vertical heat flux is examined to second order, and the result is compared with ordinary porous convection due to uniform heating from below.

2. Governing equations.

Consider natural three-dimensional convection in a porous medium which, for example, may be composed of closely packed grains, completely surrounded by a homogeneous fluid. The medium is bounded horizontally by two impermeable planes separated by a distance h, which is assumed to be small compared to the characteristic horizontal dimensions. As in [13] the boundaris are taken to be perfect heat conductors, and to have a linear temperature variation in the x-direction, see fig.1. For a given x-coordinate the temperature difference between the planes is constant, ΔT, and the lower plane is the warmer.

We introduce dimensionless variables by choosing

$$(c_p\rho)_m h^2/\lambda_m, \quad \kappa_m/h, \ \Delta T, \quad \rho_o \nu\kappa_m/K \tag{2.1}$$

as units of time, velocity, temperature and pressure, respectively.

Making the Boussinesq approximation, the governing equations may be written

$$\nabla p + \vec{v} - \mathrm{Ra}T\,\vec{j} = 0 \tag{2.2}$$

$$\nabla\cdot\vec{v} = 0 \tag{2.3}$$

$$\partial T/\partial t + \vec{v}\cdot\nabla T - \nabla^2 T = 0 \tag{2.4}$$

For details concerning the derivation of the heat eauation in a porous medium, we refer to Katto and Masuoka [15].

The system (2.2-2.4) permits a particular, steady solution. Setting

$$\partial/\partial t = v = w = 0$$

$$u = U(y), \qquad T = T(y) - \beta x \tag{2.5}$$

where β now is dimensionless, the governing equations reduce to

$$\begin{aligned} DU(y) &= \beta \mathrm{Ra} \\ D^2T(y) &= -\beta U(y) \end{aligned} \tag{2.6}$$

Here $D = d/dy$.

In a porous medium we have no restriction on the tangential velocity at a rigid boundary. However, the mass must be conserved, and hence

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} U(y)dy = 0 \tag{2.7}$$

For the temperature at the boundaries we must require

$$T(\pm\tfrac{1}{2}) = \mp\tfrac{1}{2} \tag{2.8}$$

The solution of (2.6-2.8) is easily obtained, being

$$\begin{aligned} U(y) &= \beta \mathrm{Ra}\, y \\ T(y) &= -y + \tfrac{1}{6}\beta^2 \mathrm{Ra}(\tfrac{1}{4}y - y^3) \end{aligned} \tag{2.9}$$

We remark that this solution is valid in the region where the effects of the lateral side-walls can be neglected.

Formally β and Ra are independent parameters. It is obvious, however, that the solution (2.9) is not stable for all values of these parameters. For example, when Ra is sufficiently increased, convection will occur, and a secondary flow develops. However,there is also another point which should not be overlooked. It is well known that for Darcys law to be valid in its present form, the (particle) Reynolds number should not exceed unity.

We define a Reynolds number

$$\mathrm{Re} = \frac{U_{max} d}{\nu} \tag{2.10}$$

where d is the characteristic (dimensional) grain diameter. Substituting $U_{max} = \beta \mathrm{Ra}\kappa_m/2h$, we get as a necessary condition for (2.9) to be valid that

$$\beta \mathrm{Ra} < 2\, \mathrm{Pr}(\tfrac{d}{h})^{-1} \tag{2.11}$$

where Pr is the Prandtl number.

For experimental verification this condition must be kept in mind.

The basic state given by (2.9) involves vertical as well as horizontal heat transfer. The vertical heat flux per unit area through the bottom (and the top) plane may be stated as

$$H_v = - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left(\frac{\partial T}{\partial y}\right)_{y=-\frac{1}{2}} dx = 1 + \frac{1}{12} \beta^2 Ra \tag{2.12}$$

We notice that the presence of β increases the heat flux.

The total horizontal heat flux is the same through all vertical planes, and is given by

$$H_h = - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial T}{\partial x} dy + \int_{-\frac{1}{2}}^{+\frac{1}{2}} U\,T\, dy = \beta - \frac{1}{12} \beta Ra + \frac{1}{720} \beta^3 Ra^2 \tag{2.13}$$

The solution (2.9) is asymptotically valid, i.e. when the ratio of the depth to the length approaches zero. Analogously to Batchelor [16], we may derive a more general expression for the vertical heat flux, taking the side-walls into account.

Consider a two-dimensional porous horizontal layer with dimensionless length $2(l_1 + l_2)$, and where $l_1 \gg 1$. We need to consider only the right half of it, since exactly the same arguments can be applied to the left. Suppose that the asymptotic solution (2.9) is valid for $x < l_1$. In the region $l_1 < x < l_1 + l_2$ we do not know the solution. We do know, however, that there is a horizontal flux of heat, H_h, into this part.Let us for simpliciy take the side-walls to be insulating. Then the horizontal heat flux into the end region must be removed by conduction through the horizontal boundaries in this part. Take the fraction of H_h conducted through the colder (upper) boundary to be δ. Then $1-\delta$

must be conducted through the warmer boundary. The total vertical heat transport can now be stated as

$$\mathcal{H} = 2\left[H_v l_1 + l_2 + H_h(2\delta-1)l_2\right] \tag{2.14}$$

Substituting for H_v and H_h we may write the vertical heat flux per unit area

$$H = \frac{\mathcal{H}}{2(l_1+l_2)} = 1 + \frac{\beta^2 Ra\, l_1}{12(l_1+l_2)} + \left(\beta - \frac{1}{12}\beta Ra + \frac{\beta^3 Ra^2}{720}\right)\frac{(2\delta-1)l_2}{(l_1+l_2)} \tag{2.15}$$

δ is itself a function of β and Ra, which we do not know. However, a little information can be obtained. When the motion is slow, the heat loss through the horizontal boundaries in the end regions will be nearly symmetrical, and δ will be slightly above $\frac{1}{2}$. By increasing the value of βRa, more warm fluid will accumulate in the neighbourhood of the cold boundary in the region $l_1 < x < l_1 + l_2$, and the reverse in the left part of the slot. The main heat exchange through the boundaries will occur where the temperature gradients are largest, and hence δ may finally approach unity.

When $l_1 \to \infty$ and l_2 remains finite, the heat flux reduces to (2.12).

3. Perturbation analysis.

Throughout the rest of this paper we shall assume that effects due to lateral side-walls can be neglected.

Perturbating the velocity, temperature and pressure fields, the resulting field variables may be written

$$\vec{v} = U(y)\vec{i} + \hat{\vec{v}}(x,y,z,t)$$
$$\theta = T(y) - \beta x + \hat{\theta}(x,y,z,t) \tag{3.1}$$
$$p = P(x,y) + \hat{p}(x,y,z,t)$$

where $P(x,y)$ is pressure in the basic flow.

From (2.2) we obtain

$$\nabla p + \vec{v} - Ra\theta\vec{j} = 0 \tag{3.2}$$

where the carets habe been dropped. We observe that $\vec{j}\cdot(\nabla x\vec{v}) = 0$. Since we also have $\nabla\cdot\vec{v} = 0$, the velocity is a poloidal vector and can be expressed by a single scalar function ψ as

$$\vec{v} = \nabla x(\nabla x\vec{j}\psi) \equiv \Delta\psi \tag{3.3}$$

or explicitly

$$\{u,v,w\} = \{\psi_{xy}, -\nabla_1{}^2\psi, \psi_{yz}\} \tag{3.4}$$

where $\nabla_1{}^2$ is the two-dimensional Laplacian.

From (3.2) the perturbation temperature is given by

$$\theta = -\frac{1}{Ra}\nabla^2\psi \tag{3.5}$$

Introducing ψ into the heat equation, we finally obtain

$$\nabla^4\psi + Ra\nabla_1{}^2\psi = \nabla^2\psi_t + \beta Ra[\overline{U}\nabla^2\psi_x + \psi_{xy}] + \beta^2 Ra^2 D\overline{\Theta}\nabla_1{}^2\psi + \Delta\psi\cdot\nabla\nabla^2\psi \tag{3.6}$$

where the operator Δ is defined by (3.3), and the boundary conditions being that

$$\psi = \nabla^2\psi = 0 \qquad \text{for} \quad y = \pm\tfrac{1}{2} \tag{3.7}$$

Further we have defined

$$\overline{U} \equiv U(y)/\beta Ra = y$$
$$\overline{\Theta} \equiv (T(y)+y)/\beta^2 Ra = \tfrac{1}{6}(\tfrac{1}{4}y - y^3) \tag{3.8}$$

For $\beta = 0$, (3.6) reduces to the equation for ordinary Bénard convection in a porous medium, a problem which is well known. The inclusion of a horizontal temperature gradient, however, complicates the problem considerably. In the present paper we shall therefore restrict ourselves by assuming that β is a small parameter. As usually in problems of this type, we consider infinitesimal perturbations. Neglecting terms of order ψ^2 in (3.6), and introducing

$$\psi = \tilde{\psi}(y)\exp(i(kx+mz)+\sigma t) \tag{3.9}$$

where k and m are real wave numbers in the x- and z-direction, respectively, and $\sigma = \sigma^r + i\sigma^i$ is the complex growth rate, the perturbation equation may be written

$$\{(D^2-\alpha^2)^2-\alpha^2 Ra^c\}\tilde{\psi} = \sigma(D^2-\alpha^2)\tilde{\psi} + ik\beta Ra^c\{\bar{U}(D^2-\alpha^2)+D\}\tilde{\psi} - (\alpha\beta Ra^c)^2 D\bar{\Theta}\tilde{\psi} \tag{3.10}$$

to be solved subject to

$$\tilde{\psi} = D^2\tilde{\psi} = 0 \qquad \text{for } y = \pm \tfrac{1}{2} \tag{3.11}$$

Here α is the horizontal overall wave number defined by $\alpha^2 = k^2 + m^2$, and Ra^c the critical Rayleigh number corresponding to the onset of convection.

The solutions will be obtained by a series expansion after β as a small parameter, as in [13]. This procedure is analogous to those previously applied in [17] for convection in Couette flow and in [18] for convection in a tilted slot.

We introduce the series expansions

$$\tilde{\psi} = \sum_{n=o}^{\infty} \beta^n \tilde{\psi}_n, \quad Ra^c = \sum_{n=o}^{\infty} \beta^n R_n, \quad k = \sum_{n=o}^{\infty} \beta^n k_n,$$

$$m = \sum_{n=o}^{\infty} \beta^n m_n, \quad \sigma = \sum_{n=o}^{\infty} \beta^n \sigma_n \tag{3.12}$$

By substituting these expansions into (3.10) and equating equal powers of β, an infinite set of inhomogeneous differential equations is obtained. $R_0, R_1, R_2, \cdots$ are found from the solvability conditions for these equations, and the wave number terms $k_0, m_0, k_1, m_1 \cdots$ are determined so that they minimize the critical Rayleigh number.

We may do some preliminary simplifying observations. Changing the sign of β in (2.5) merely leads to a reverse of the direction of the basic flow. Physically this cannot alter the stability conditions, i.e. the critical Rayleigh number and the corresponding wave number. Hence Ra^c,k and m should not contain odd powers of β, or

$$R_{2i+1} = k_{2i+1} = m_{2i+1} = 0 , \qquad i = 0,1,2,\cdots \tag{3.13}$$

We consider the transition from stable to unstable solutions. This transition goes through a neutral state, characterized by $\sigma^r = 0$. Generally we can not prove that the principle of exchanges of stabilities (PES) is valid, i.e. that the neutrally stable solutions are stationary. However, when β is small enough for the series (3.12) to converge, this can be proved. For the zero-order system ($\beta = 0$), PES is obviously valid, implying $\sigma_o^i = 0$. Further the solution must be even, since the boundary conditions are. Owing to the uneven character of the operator $(\overline{U}(D^2-\alpha^2) + D)$

appearing on the right of (3.10), and the fact that Ra^c must be real, we immediately obtain from the solvability condition in the following orders that $\sigma_1{}^i = 0$, $\sigma_2{}^i = 0$ and so on. Hence oscillatory instability does not occur, and we may put $\sigma = 0$ in (3.10).

The zeroth-order system corresponds to convection without shear. The equation reduces to

$$L\tilde{\psi}_o \equiv \{(D^2-\alpha_o{}^2)^2 - \alpha_o{}^2R_o\}\tilde{\psi}_o = 0 \tag{3.14}$$

with $\tilde{\psi}_o = D^2\tilde{\psi}_o = 0,\ y = \pm\frac{1}{2}$

The solution may be written

$$\tilde{\psi}_o = A\cos\pi y \tag{3.15}$$

The minimum Rayleigh number is found to be

$$R_o = 4\pi^2 \quad \text{with} \quad \alpha_o{}^2 = k_o{}^2 + m_o{}^2 = \pi^2 \tag{3.16}$$

The zeroth-order system is easily shown to be self-adjoint. Hence the condition for the higher order equations to have a non-trivial solution may be stated as

$$\langle\tilde{\psi}_o L\,\tilde{\psi}_n\rangle = 0\ , \qquad n = 1,2,3,--- \tag{3.17}$$

where the brackets denote integration from $y = -\frac{1}{2}$ to $y = +\frac{1}{2}$. From (3.10) the first-order system is given by

$$\begin{aligned} L\tilde{\psi}_1 &= ik_oR_o[\overline{U}(D^2-\alpha_o{}^2) + D]\tilde{\psi}_o \\ &= -i4\pi^3Ak_o[2\pi y\cos\pi y + \sin\pi y] \end{aligned} \tag{3.18}$$

subject to $\tilde{\psi}_1 = D^2\tilde{\psi}_1 = 0$ for $y = \pm\frac{1}{2}$.

We observe that the solvability condition is identically satisfied.

In order to avoid the arbitrary homogeneous solution which always can be added in each order, we choose as a normalization condition

$$\langle \tilde{\psi}_o \tilde{\psi} \rangle = \tfrac{1}{2} \tag{3.19}$$

Hence $A = 1$.

By noting that the operator $(D^2 + \pi^2)^2$ annihilates the right hand side of (3.18), the evaluation is straight foreward, giving

$$\tilde{\psi}_1 = i\frac{k_o}{2}[-\frac{\pi}{4} \sin\pi y + y\cos\pi y + \pi y^2 \sin\pi y] \tag{3.20}$$

Next, to second order, we get

$$\begin{aligned} L\tilde{\psi}_2 = {} & [\alpha_2{}^2(2(D^2-\alpha_o{}^2) + R_o) + \alpha_o{}^2 R_2 - \alpha_o{}^2 R_o{}^2 D\bar{\Theta}]\tilde{\psi}_o \\ & + ik_o R_o[\bar{U}(D^2-\alpha_o{}^2) + D]\tilde{\psi}_1 \end{aligned} \tag{3.21}$$

where $$\alpha_2{}^2 = 2(k_o k_2 + m_o m_2) \tag{3.22}$$

Applying the solvability condition (3.17), we finally obtain

$$R_2 = 4\pi^2 + 3\, k_o{}^2 \tag{3.23}$$

Thus we observe that a disturbance given by $k_o = 0$, and hence $m_o = \pi$, minimizes R_2. This particular disturbance defines a longitudinal roll. Then, in a physical problem, as the critical Rayleigh number is approached from below, a longitudinal roll first starts to grow exponentially. Accordingly it constitutes the preferred mode among the infinite number initially present.

Unfortunately, the first term on the right hand side in (3.21), being proportional to $\alpha_2{}^2$, vanishes identically. We therefore must proceed to fourth order to obtain a correction on the critical wave number.

Substituting (3.23) into (3.21), we may calculate $\tilde{\psi}_2$, which is an elementary, but lengthy task. The equation and the result are given in the appendix.

Since we already have shown that longitudinal rolls will be preferred, it is physically relevant to put $k_o = 0$ in the remaining analysis. This means $\tilde{\psi}_1 = 0$. The third-order equation then reduces to that previously derived in first order when substituting k_2 for k_o. Accordingly the solution may be written

$$\tilde{\psi}_3 = \frac{ik_2}{2}[-\frac{\pi}{4}\sin\pi y + y\cos\pi y + \pi y^2\sin\pi y] \tag{3.24}$$

For the fourth-order system we obtain

$$L\tilde{\psi}_4 = [2\alpha_4{}^2(D^2-\alpha_o{}^2)+\alpha_4{}^2R_o-\alpha_2{}^4+\alpha_2{}^2R_2+\alpha_o{}^2R_4-\alpha_2{}^2R_o{}^2D\overline{\Theta} - 2\alpha_o{}^2R_oR_2D\overline{\Theta}]\tilde{\psi}_o+[2\alpha_2{}^2(D^2-\alpha_o{}^2)+\alpha_2{}^2R_o+\alpha_o{}^2R_2-\alpha_o{}^2R_o{}^2\, D\overline{\Theta}]\tilde{\psi}_2 \tag{3.25}$$

where now $\alpha_o{}^2 = m_o{}^2 = \pi^2$, $\alpha_2{}^2 = 2m_om_2$, $\alpha_4{}^2 = k_2{}^2 + m_2{}^2 + 2m_om_4$

and $R_2 = R_o = 4\pi^2$. (3.26)

Applying the solvability condition

$$\langle\tilde{\psi}_oL\tilde{\psi}_4\rangle = 0 \tag{3.27}$$

we finally get

$$R_4 = 4m_2{}^2 + R_o{}^2[\frac{\pi^2}{360} + \frac{1}{16\pi^2}(31 - \frac{7\pi^2}{3}) - \frac{3}{16\sqrt{3}\pi}\,\mathrm{tgh}(\pi\sqrt{3}/2)] \tag{3.28}$$

From this it follows that R_4 has a minimum for $m_2 = 0$. Accordingly, the critical Rayleigh number to fourth order may be written

$$Ra^c = 4\pi^2(1 + \beta^2 + 1.73\beta^4 + + \cdots) \tag{3.29}$$

and the critical wave numbers

$$\begin{aligned} k &= O(\beta^2) \\ m &= \pi + O(\beta^4) \end{aligned} \tag{3.30}$$

We observe that Ra^c is always larger than for ordinary convection in a porous medium. Physically this is due to the presence of warm fluid above cold fluid in the basic flow.

4. Energy considerations.

In order to gain some physical insight into why longitudinal rolls should be preferred, we consider the equation for the kinetic energy of the perturbation. As mentioned in the introduction, shear instabilities do not occur in a porous medium owing to the lack of inertial terms in the equation for momentum transfer. The mechanism selecting the preferred mode must then be purely thermal.

Taking the real part of (3.2), multiplying by the real part of $\vec{v}$, averaging over a wave length in the x- and z-directions, and integrating from $y = -\frac{1}{2}$ to $y = +\frac{1}{2}$, using the boundary conditions, we readily obtain

$$<\overline{\vec{v}^2}> = Ra<\overline{v\theta}> \tag{4.1}$$

where the bar and the brackets denote mean and vertical integration, respectively.

This equation expresses a balance in the perturbation energy between the gain from potential energy and the loss by the viscous dissipation. In a porous medium, however, the latter is directly proportional to the averaged kinetic energy of the perturbation. Hence we may write

$$KE \equiv \tfrac{1}{2}\langle\overline{\vec{v}^2}\rangle = \tfrac{1}{2}\, Ra\langle\overline{v\theta}\rangle = \tfrac{1}{2}\langle\overline{\nabla_1{}^2\psi\nabla^2\psi}\rangle \tag{4.2}$$

where we have substituted from (3.4) and (3.5).

To second order in the marginal stable solutions, the above expression reduces to

$$KE = \frac{\pi^4}{4} - \beta^2\,\frac{\pi^2}{4}\,\langle\tilde{\psi}_1{}^i(D^2-\pi^2)\tilde{\psi}_1{}^i\rangle \tag{4.3}$$

From (3.20) it follows that $\tilde{\psi}_1{}^i$ may be written

$$\tilde{\psi}_1{}^i = k_o F(y) \qquad \text{where} \qquad F(\pm\tfrac{1}{2}) = D^2F(\pm\tfrac{1}{2}) = 0 \tag{4.4}$$

Accordingly

$$KE = \frac{\pi^4}{4} + \beta^2\,\frac{\pi^2}{4}\,k_o{}^2\,\langle(DF)^2 + \pi^2F^2\rangle \tag{4.5}$$

The last term is obviously positive. Hence we may conclude that, among all marginally stable solutions, the preferred mode ($k_o = 0$) will have minimum kinetic energy (or, more precisely, minimum dissipation). Since KE is directly proportional to the released potential energy, we further conclude that the preferred mode is characterized by minimum potential energy. Equivalently, that particular mode which involves least possible energy conversion, will be selected.

5. Finite Amplitude solution.

In the previous sections we have demonstrated that a preferred mode of disturbance is predicted from linear theory. Since this particular disturbance is the fastest growing, it also will dominate the motion at slightly supercritical Rayleigh numbers, suppressing the growth of other unstable modes in this region. Accordingly, we look for a stationary solution of the nonlinear problem considering longitudinal modes only.

Setting $\partial/\partial t = \partial/\partial x = 0$, (3.6) reduces to

$$\nabla^4\psi + Ra\nabla_1{}^2\psi = \beta^2 Ra^2 D\overline{\Theta}\nabla_1{}^2\psi + \Delta\psi\cdot\nabla\nabla^2\psi \tag{5.1}$$

This equation will be solved by a two-parameter expansion, and the solution may be written

$$\psi = \sum_{m=1,n=o}^{\infty} \varepsilon^m \beta^n \psi^{(mn)} \tag{5.2}$$

provided the series converge . Since β appears only as squared in (5.1), the summation can be taken over even n. The parameter ε will be defined by

$$\varepsilon^2 = \frac{Ra-Ra^c}{Ra} \tag{5.3}$$

which is analogous to the definition originally proposed by Kuo [19] for a similar problem. In the present case, however, Ra^c is a function of β, given by (3.29). We note that ε is always less than one.

Equation (5.3) may also be written

$$Ra = \frac{Ra^c}{1-\varepsilon^2} = Ra^c + Ra_s^c(\varepsilon^2+\varepsilon^4+ \cdots + \varepsilon^{2s}) \tag{5.4}$$

where $$Ra_s^c = Ra^c\Big/(1-\varepsilon^{2s}) \tag{5.5}$$

When solving to second order, we choose $s = 1$, to fourth order $s = 2$ and so on. By writing Ra as a "finite" sum, we are, to every order, working with a correct Rayleigh number. It appears that this procedure highly improves the convergence of the solution. (Kuo [19], Palm et al. [5]).

Substituting the expansions (5.2) and (5.4) into (5.1) and using ε and β as ordering parameters, we obtain an infinite set of equations. In this procedure ε and β appear as given small parameters. Expanding the amplitude A of the solution after ε and β, the A_{mn} will be determined at each order so as to satisfy the solvability conditions.

To order ε^1, β^0 the y-dependence of the solution is given by (3.15). For a longitudinal roll we then write

$$\psi^{(10)} = A_{10}\cos\pi y\cos\pi z \tag{5.6}$$

where we have chosen $m = \pi$, since this is the physically relevant wave number for $Ra > Ra^c$.

Order ε^2, β^0 is given by

$$(\nabla^4 + R_o\nabla_1{}^2)\psi^{(20)} = \Delta\psi^{(10)}\circ\nabla\nabla^2\psi^{(10)} \tag{5.7}$$

The solution is easily obtained, being

$$\psi^{(20)} = A_{20}\cos\pi y\cos\pi z + \frac{\pi A_{10}^2}{16}\sin 2\pi y \tag{5.8}$$

The result to order ε^6, β^0 has in fact been computed in [5].

The unknown amplitudes are determined from the solvability condition, giving

$$A_{10} = \frac{4}{\pi}\left(\frac{R_{os}}{R_o}\right)^{\frac{1}{2}}, \qquad A_{20} = 0 \tag{5.9}$$

where $R_{os} = R_o\big/(1-\varepsilon^{2s})$.

In the present paper we study the change of the vertical heat transport due to the inclusion of a small horizontal temperature gradient. By averaging the stationary heat equation (2.4) and utilizing that $\vec{v}$ is periodic i.e. $\overline{\vec{v}} = 0$, we obtain by integration

$$H' = - D\overline{\theta} + \overline{v\theta} \,, \tag{5.10}$$

independent of y. The total vertical heat flux may then be written

$$H = H_V + H' = 1 + \frac{1}{12}\beta^2 Ra + \frac{1}{Ra}(D^3\overline{\psi})_{y=-\frac{1}{2}} \tag{5.11}$$

where we have substituted for H_V and θ from (2.12) and (3.5), respectively.

In [5] the Nusselt number (which corresponds to $H(\beta=0)$) has been obtained to sixth order. However, when the horizontal temperature dependence is taken into account, the inhomogeneous differential equations derived in each order, very soon become unsuitable for analytical treatment. We shall therefore not push the computations further than necessary to obtain a correction on the second-order heat flux. To achieve this goal, we must solve the system of equations to order ε^3, β^2.

To order ε^1, β^2 the equation is given by (3.21). Including the z-dependence and adding a homogeneous solution, we may write

$$\psi^{(12)} = A_{12}\cos\pi y\cos\pi z + A_{10}\tilde{\psi}_2\cos\pi z \tag{5.12}$$

where $\tilde{\psi}_2$ is given in the appendix.

Derivation of the equation to order ε^2, β^2, yields

$$(\nabla^4+R_o\nabla_1{}^2)\psi^{(22)}=(-R_2+R_o{}^2D\bar{\Theta})\nabla_1{}^2\psi^{(20)} + \Delta\psi^{(10)}\cdot\nabla\nabla^2\psi^{(12)} + \Delta\psi^{(12)}\cdot\nabla\nabla^2\psi^{(10)} \tag{5.13}$$

Having in mind the large number of terms in $\psi^{(12)}$, the evaluation of $\psi^{(22)}$ from (5.13) obviously is a long and tedious task. However, it follows from the results in the appendix that $\tilde{\psi}_2$ may be approximated within a few percent by the first term in a series expansion $\sum_{n=1}^{N} c_{2n+1}\cos(2n+1)\pi y$. It turns out that $c_3 = -\frac{1}{32}$. Accordingly we take

$$\psi^{(12)} = A_{12}\cos\pi y\cos\pi z - \frac{A_{10}}{32}\cos3\pi y\cos\pi z \tag{5.14}$$

in the following analysis.

By substituting (5.14) into (5.13), we get

$$\psi^{(22)} = A_{22}\cos\pi y\cos\pi z + \frac{\pi}{8}(A_{10}A_{12}-\frac{A_{10}{}^2}{32})\sin2\pi y - \frac{\pi A_{10}{}^2}{128}[\frac{1}{4}\sin4\pi y+\frac{1}{3}\sin2\pi y\cos2\pi z+\frac{1}{12}\sin4\pi y\cos2\pi z] \tag{5.15}$$

where we have utilized that $A_{20} = 0$.

Proceeding to order ε^3, β^2, the equation is given by

$$(\nabla^4+R_o\nabla_1{}^2)\psi^{(32)} = (-R_{2s}+2R_oR_{os}D\bar{\Theta})\nabla_1{}^2\psi^{(10)}-R_{os}\nabla_1{}^2\psi^{(12)} + (-R_2+R_o{}^2D\bar{\Theta})\nabla_1{}^2\psi^{(30)}+\Delta\psi^{(10)}\cdot\nabla\nabla^2\psi^{(22)} + \Delta\psi^{(22)}\cdot\nabla\nabla^2\psi^{(10)}+\Delta\psi^{(12)}\cdot\nabla\nabla^2\psi^{(20)}+\Delta\psi^{(20)}\cdot\nabla\nabla^2\psi^{(12)} \tag{5.16}$$

where $R_{2s} = R_{os}$.

Applying the solvability condition to this equation, we finally obtain

$$A_{12} = -\frac{7}{16} A_{10} \tag{5.17}$$

The fact that A_{10} and A_{12} have opposite sign, means that the horizontal temperature gradient acts to diminish the magnitude of the velocity. Physically this is due to the stabilizing configuration in the basic flow, where the average density in the upper part is less than in the lower.

The perturbation heat flux to this order may now be written

$$H' = \frac{1}{Ra}\left[\varepsilon^2 D^3 \overline{\psi}^{(20)} + \varepsilon^2 \beta^2 D^3 \overline{\psi}^{(22)}\right]_{y=-\frac{1}{2}} \tag{5.18}$$

Substitution from (5.9), (5.15) and (5.17) yields

$$H' = 2\left(\frac{R_{os}}{Ra}\right)\left(1 - \frac{11}{16}\beta^2\right)\varepsilon^2 \tag{5.19}$$

From this we observe that the presence of β reduces the perturbation heat transport. However, by inspecting the total vertical heat flux (5.11), we find that this reduction is more than compensated for by the flux conducted through the boundaries. This becomes clear from Fig.2 where $H = H_V + H'$ is plotted against Ra for various values of β. We then conclude that the introduction of a small horizontal temperature gradient into the classic Bénard problem leads to an increase of the total vertical heat flux. In the calculation of H' we have used $s = 1$, which gives the best approximation to this order. The breaks in the slope of the heat transport curves in Fig. 2 indicate when convection commences, and the figure clearly exhibits the stabilizing effect of β, as mentioned in section 3.

Finally we remark that the total horizontal heat flux remains that of the basic state (2.13), since the considered disturbance is independent of x.

6. Summary and concluding remarks.

According to the results presented above, the Rayleigh number at the neutral state will have a minimum value for steady longitudinal rolls with axes aligned in the direction of the basic flow. The critical Rayleigh number will always be larger than that corresponding to convection with uniform heating from below. These conclusions are similar to those reached in [13] for a viscous fluid in the limit of infinite Prandtl number.

The instability is of thermal origin, and among the marginally stable solutions the preferred mode has minimum potential energy.

Assuming that the initially preferred mode also dominates at moderately supercritical Rayleigh numbers, a stationary finite amplitude solution is obtained. The vertical heat flux is examined to second order, and a small horizontal temperature gradient β is found to diminish the vertical perturbation heat transport. The heat conducted through the boundaries in the vertical direction is increased, however, so the total vertical heat flux is an increasing function of β.

Before closing, we note that by working with supercritical Rayleigh numbers, rolls having axes tilted a small angle to the basic flow become linearly unstable. Such modes may be considered as perturbations to our stationary solution. When β is zero, it can be shown analogously to [18] that the stationary roll is stable. For non-zero β the stability problem becomes more complex, and should probably be attacked numerically. This will be left for future work, however.

Finally we remark that the inclusion of lateral side-walls will strongly influence the selection of mode. Presumably an increase of the aspect ratio (height to length) should favour transverse rolls, i.e. rolls with axes normal to the basic flow.

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APPENDIX

Equation (3.21) can be written

$$L\tilde{\psi}_2 = (4\pi^4 - \tfrac{2}{3}\pi^6)\cos\pi y + 8\pi^6 y^2\cos\pi y + k_o{}^2\pi^2[(1+\frac{\pi^2}{2})\cos\pi y \qquad (A.1)$$

$$- 2\pi(1+\frac{\pi^2}{2})\, y\sin\pi y - 6\pi^2 y^2\cos\pi y + 4\pi^3 y^3\sin\pi y]$$

with boundary conditions $\tilde{\psi}_2 = D^2\tilde{\psi}_2 = 0$ for $y = \pm\frac{1}{2}$. By noting that the term on the right is annihilated by the operator $(D^2+\pi^2)^4$, the solution is obtained by straight foreward computation, being

$$\tilde{\psi}_2 = a_1\cos\pi y + a_2\cosh(\pi\sqrt{3}y) + a_3 y\sin\pi y + a_4 y^2\cos\pi y$$

$$+ a_5 y^3\sin\pi y + a_6 y^4\cos\pi y \qquad (A.2)$$

where

$$a_1 = \frac{\pi^2}{12} - \frac{3}{4} + \frac{k_o{}^2}{16}(\frac{7\pi^2}{120} + \frac{1}{6} - \frac{7}{\pi^2}); \qquad a_2 = -(\frac{\pi}{4} - \frac{k_o{}^2}{8\pi})\Big/\cosh(\pi\sqrt{3}/2)$$

$$a_3 = \frac{\pi}{2} + \frac{\pi^3}{12} - \frac{k_o{}^2}{8\pi}(2 - \frac{\pi^2}{2}); \qquad a_4 = -\pi^2 - \frac{k_o{}^2}{16}(2+\pi^2);$$

$$a_5 = -\frac{\pi^3}{3} - \frac{k_o{}^2\pi}{4}; \qquad a_6 = \frac{k_o{}^2\pi^2}{8}.$$

For longitudinal rolls, i.e. $k_o = 0$, (A.1) has been solved by Galerkins' method. Since the inhomogeneous term is even and zero at $y = \pm\frac{1}{2}$, the solution may be written

$$\tilde{\psi}_2 = \sum_{n=1}^{N} \frac{(-1)^n(2n+1)\cos(2n+1)\pi y}{4n^3(n+1)^3(n^2+n+1)} \qquad (A.3)$$

In the table below is listed the exact solution (A.2) for longitudinal rolls at various values of y, and the approximations by taking $N = 1$ and $N = 2$ in (A.3).

Table 1. Values of $\tilde{\psi}_2$.

y	0	0•1	0•2	0•3	0•4	0•5
$10^2 \times \tilde{\psi}_2$ (exact)	-3•05	-1•83	0•89	2•96	2•62	0
$10^2 \times \tilde{\psi}_2$ $(N = 1)$	-3•13	-1•84	0•96	2•97	2•53	0
$10^2 \times \tilde{\psi}_2$ $(N = 2)$	-3•04	-1•84	0•88	2•97	2•61	0

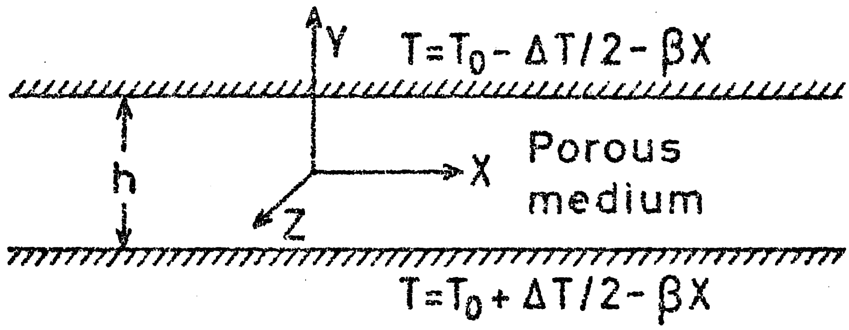

Fig.1. Temperature distribution in the model.
β is a positive constant

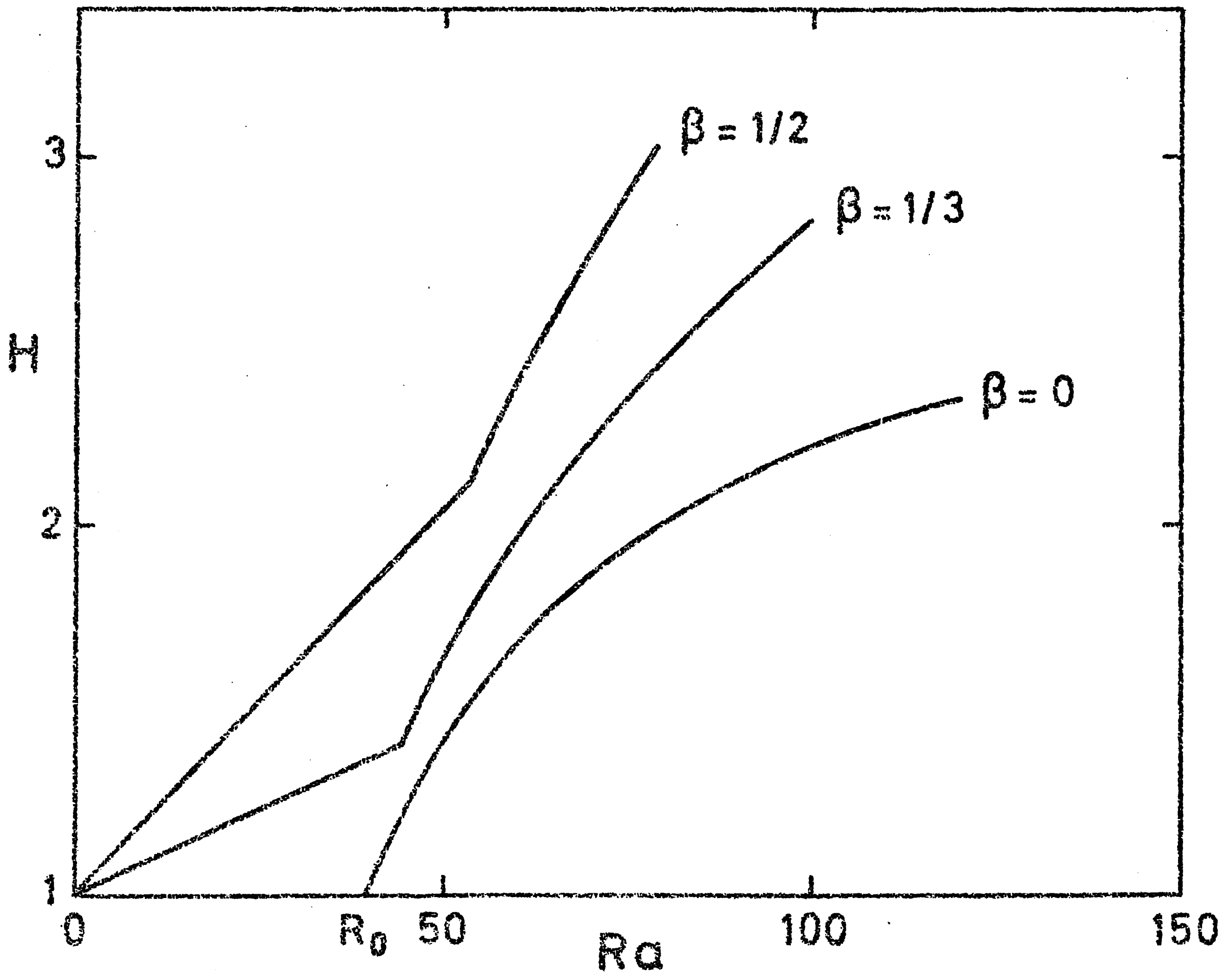

Fig.2. The total vertical heat flux H vs. Ra for various values of (dimensionless) β.

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